UF | UNIVERSITY of FLORIDA

Pre-Health Post-Baccalaureate Program
Study Guide and Practice Problems

Course: MAC2311

Textbook Chapter: 3.11 - 4.1 (Rogawski 2e)

Topics Covered: Related rates
Linear approximation

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Related Rates

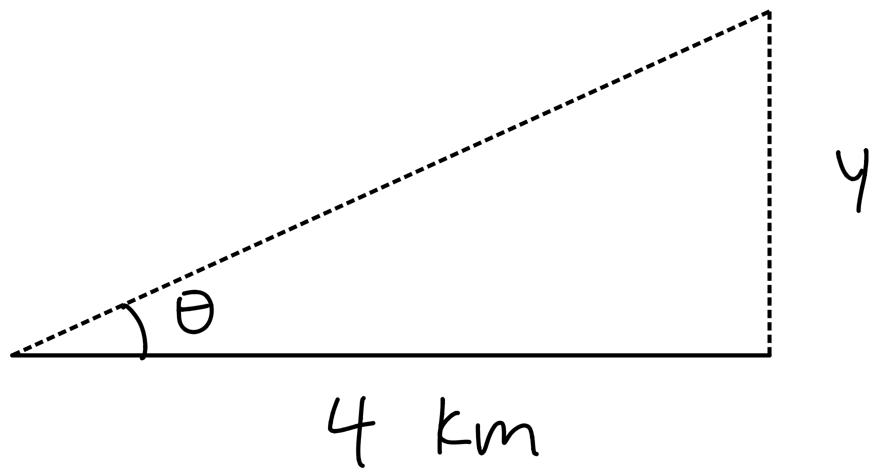
- These problems don't really require any new Calculus knowledge, however, we do need to talk about some problem solving strategies.
- What is the goal of a related rates problem?
  In these problems, we are trying to relate the known change of one variable with the unknown change in another variable.
- To connect the two, we will need to form an equation. This will usually be a common equation, such as the

pythagorean theorem, formula of a circle, equation for the volume of a box, etc.

- The best way to demonstrate how to solve these problems is to work through one:

<u>Question</u>: A group of people watch a spaceship launch from KSC, standing 4 km from the launch site. When the angle between the viewers' eyes and the ground is $\pi/3$, the angle is changing at a rate of 1.5 rad/min. What is the spaceship's velocity at this moment?

① Draw a picture



② List what we know

$$\frac{d\theta}{dt} = 1.5 \text{ rad/min}$$

when $\theta = \frac{\pi}{3}$

$$\frac{dy}{dt} = \text{velocity}$$

③ Create an equation that relates θ with y

$$\tan\theta = \frac{y}{4}$$

④ Differentiate with respect to one variable

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{4} \frac{dy}{dt}$$

$$\frac{dy}{dt} = 4\sec^2\theta \frac{d\theta}{dt}$$

$$\frac{dy}{dt} = \frac{4}{\cos^2\theta} \frac{d\theta}{dt}$$

⑤ Plug in

$$\frac{dy}{dt} = \frac{4}{\cos^2\left(\frac{\pi}{3}\right)} \cdot 1.5$$

$$\frac{dy}{dt} = \text{velocity} = 24 \text{ km/min}$$

Linear Approximation

- Let's go back to the beginning to the definition of the derivative:

$$\Delta f = f(a + \Delta x) - f(a)$$

- Linear approximation allows us to approximate the value of Δf, assuming that the value of Δx is very small (or, essentially zero):

$$\Delta f = f(a + \cancelto{0}{\Delta x}) - f(a)$$

$$\Delta f = f(a) - f(a)$$

$$\approx \Delta f = f'(a)\Delta x$$

- In differential notation, we write this as:

$$dy = f(a + dx) - f(a)$$

$$\approx dy = f'(a)\,dx$$

- Linearization allows us to approximate the valne of the tangent line of f at $x = a$. This function is defined as L:

$$L(x) = f'(a)(x-a) + f(a)$$

such that the linear approximation is given by:

$$L(x) \approx f(x)$$

Problems

① A cone-shaped water tank with a height of 14 ft. and a base radius of 5 ft. is leaking water at a rate of 2 ft/hr. What is the rate of change when the water depth is 6 ft?

(2) Two people on bikes are separated by 350 m. The first person starts riding directly north at 5 m/s. Seven minutes later, the second person begins riding south at 3 m/s. What is the rate of change of distance between the two riders 25 minutes after the first person began riding?

③ Use linear approximation to estimate the value of $e^{0.1}$

4 Find a linear approximation to $f(x) = 3xe^{2x-10}$ at $x = 5$

Solutions

① A cone-shaped water tank with a height of 14 ft. and a base radius of 5 ft. is leaking water at a rate of 2 ft/hr. What is the rate of change when the water depth is 6 ft?

$$\frac{r}{h} = \frac{5}{14}$$

$$r = \frac{5}{14}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{5}{14}h\right)^2 h$$

$$V = \frac{1}{3}\pi \frac{5}{14}h^3$$

$$V' = \frac{25}{196}\pi h^2 \cdot h'$$

$$-2 = \frac{25}{196}\pi (36) \cdot h'$$

$$h' = -0.14$$

② Two people on bikes are separated by 350 m. The first person starts riding directly north at 5 m/s. Seven minutes later, the second person begins riding south at 3 m/s. What is the rate of change of distance between the two riders 25 minutes after the first person began riding?

$x' = 5$ m/s

$y' = 3$ m/s

$x = 5(25)(60)$
$= 7500$ m

$z^2 = (x+y)^2 + 350^2$

$2z\,dz = 2(x+y)(dx+dy)$

$dz = \dfrac{\cancel{2}(x+y)(dx+dy)}{\cancel{2}z}$

$$y = 3(18)(60)$$
$$= 3240 \text{ m}$$

$$z = \sqrt{(x+y)^2 + 350^2}$$
$$= 10745.7 \text{ m}$$

$$dz = \frac{(7500 + 3240)(5+3)}{10745.7}$$

$$dz \approx 8 \text{ m/s}$$

③ Use linear approximation to estimate the value of $e^{0.1}$

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$L(t) = 1 + (1)(x-0) = x+1$

$L(0.1) = 0.1 + 1 = 1.1$

Actual value of $e^{0.1} = 1.105$

④ Find a linear approximation to $f(x) = 3xe^{2x-10}$ at $x=5$

$f(x) = 3xe^{2x-10}$ $\qquad$ $f(5) = 15$

$f'(x) = 3e^{2x-10} + 6xe^{2x-10}$ $\qquad$ $f'(5) = 33$

$L(x) = 15 + 33(x-5)$

$L(x) = 33x - 150$